INSTITUTO FEDERAL DE EDUCAÇÃO, CIÊNCIA E TECNOLOGIA DE SANTA CATARINA
CAMPUS SÃO JOSÉ
Componente Curricular: Processos Estocásticos
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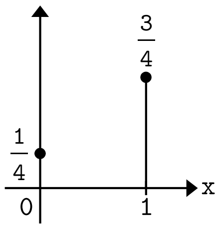
Avaliação 2

5. Sejam $B_1, B_2, B_3 \sim \text{Bern}\left(\frac{3}{4}\right)$ variáveis aleatórias sorteadas independentemente. Sejam $X = B_1 + B_2 + B_3$, $Y = B_1 + B_2 - B_3$.

a) Determine a PMF conjunta de X e Y.
b) Determine e esboce as PMFs marginais de X e Y.
c) Determine e esboce as PMFs condicionais de X dado que Y = y, para dois valores de $y \in S_y$ à sua escolha.
d) Determine a covariância entre X e Y.

Resolução:

Inicialmente, temos que B_1, B_2 e B_3 é dado pela distribuição $\text{Bern}\left(\frac{3}{4}\right)$:



Dessa forma, temos que B_1, B_2 e B_3 podem assumir valores 0 ou 1 (fracasso ou sucesso, respectivamente).

a) Determine a PMF conjunta de X e Y.

B_1	B_2	B_3	X	Y	P_r
0	0	0	0	0	$\frac{1}{4} \cdot \frac{1}{4} \cdot \frac{1}{4} = \frac{1}{64}$
0	0	1	1	-1	$\frac{1}{4} \cdot \frac{1}{4} \cdot \frac{3}{4} = \frac{3}{64}$
0	1	0	1	1	$\frac{1}{4} \cdot \frac{3}{4} \cdot \frac{1}{4} = \frac{3}{64}$
0	1	1	2	0	$\frac{1}{4} \cdot \frac{3}{4} \cdot \frac{3}{4} = \frac{9}{64}$
1	0	0	1	1	$\frac{3}{4} \cdot \frac{1}{4} \cdot \frac{1}{4} = \frac{3}{64}$
1	0	1	2	0	$\frac{3}{4} \cdot \frac{1}{4} \cdot \frac{3}{4} = \frac{9}{64}$
1	1	0	2	2	$\frac{3}{4} \cdot \frac{3}{4} \cdot \frac{1}{4} = \frac{9}{64}$
1	1	1	3	1	$\frac{3}{4} \cdot \frac{3}{4} \cdot \frac{3}{4} = \frac{27}{64}$

$P_{X,Y}(x,y)$				
	y = -1	y = 0	y = 1	y = 2
x = 0	0	$\frac{1}{64}$	0	0
x = 1	$\frac{3}{64}$	0	$\frac{3}{64} + \frac{3}{64} = \frac{3}{32}$	0
x = 2	0	$\frac{9}{64} + \frac{9}{64} = \frac{9}{32}$	0	$\frac{9}{64}$
x = 3	0	0	$\frac{27}{64}$	0

A PMF conjunta é dada por $P_{X,Y}(x,y)$, indicado na tabela ao lado.

b) Determine e esboce as PMFs marginais de X e Y.

Para $P_x(x)$, temos:

$$P_x(x = 0) = P_x(x = 0,\ y = -1) + P_x(x = 0,\ y = 0) + P_x(x = 0,\ y = 1) + P_x(x = 0,\ y = 2) = \frac{1}{64}$$

$$P_x(x = 1) = P_x(x = 1,\ y = -1) + P_x(x = 1,\ y = 0) + P_x(x = 1,\ y = 1) + P_x(x = 1,\ y = 2) = \frac{9}{64}$$

$$P_x(x = 2) = P_x(x = 2,\ y = -1) + P_x(x = 2,\ y = 0) + P_x(x = 2,\ y = 1) + P_x(x = 2,\ y = 2) = \frac{27}{64}$$

$$P_x(x = 3) = P_x(x = 3,\ y = -1) + P_x(x = 3,\ y = 0) + P_x(x = 3,\ y = 1) + P_x(x = 3,\ y = 2) = \frac{27}{64}$$

Para $P_y(y)$, temos:

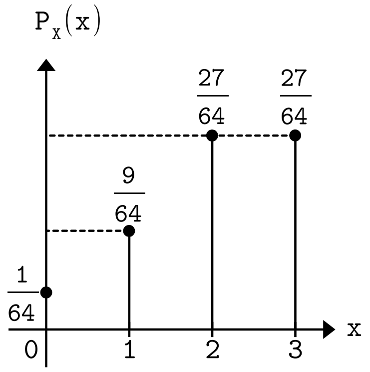
$$P_y(y = -1) = P_y(x = 0,\ y = -1) + P_x(x = 1,\ y = -1) + P_x(x = 2,\ y = -1) + P_x(x = 3,\ y = -1) = \frac{3}{64}$$

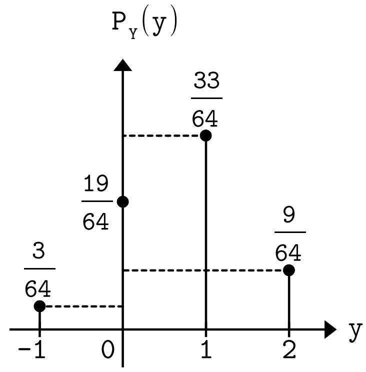
$$P_y(y = 0) = P_y(x = 0,\ y = 0) + P_x(x = 1,\ y = 0) + P_x(x = 2,\ y = 0) + P_x(x = 3,\ y = 0) = \frac{19}{64}$$

$$P_y(y = 1) = P_y(x = 0,\ y = 1) + P_x(x = 1,\ y = 1) + P_x(x = 2,\ y = 1) + P_x(x = 3,\ y = 1) = \frac{33}{64}$$

$$P_y(y = 2) = P_y(x = 0,\ y = 2) + P_x(x = 1,\ y = 2) + P_x(x = 2,\ y = 2) + P_x(x = 3,\ y = 2) = \frac{9}{64}$$

$P_{X,Y}(x,y)$					
	y = -1	y = 0	y = 1	y = 2	$P_x(x)$
x = 0	0	$\frac{1}{64}$	0	0	$\frac{1}{64}$
x = 1	$\frac{3}{64}$	0	$\frac{6}{64}$	0	$\frac{9}{64}$
x = 2	0	$\frac{18}{64}$	0	$\frac{9}{64}$	$\frac{27}{64}$
x = 3	0	0	$\frac{27}{64}$	0	$\frac{27}{64}$
$P_y(y)$	$\frac{3}{64}$	$\frac{19}{64}$	$\frac{33}{64}$	$\frac{9}{64}$	1





c) Determine e esboce as PMFs condicionais de X, dado que Y = y, para dois valores de y $\in$ S_y à sua escolha.

$$P_X(x \mid Y=y) = \frac{P_{X,Y}(x,y)}{P_Y(y)}$$

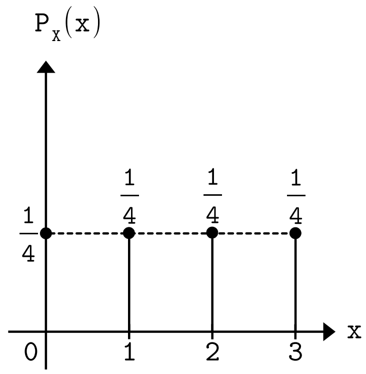
Para os valores de y <u>0</u> e <u>2</u>:

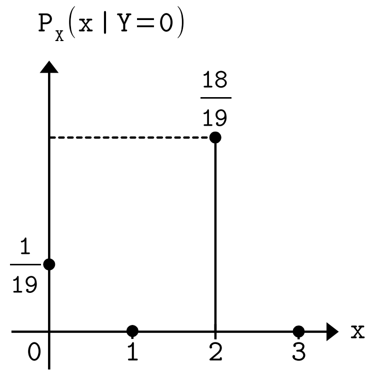
Para y = 0:

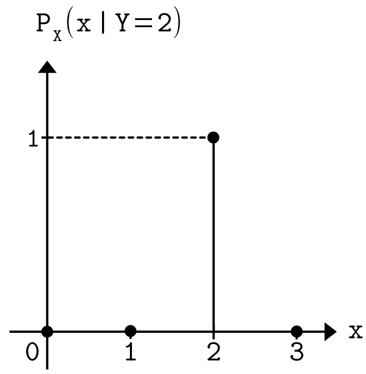
$$P_X(x \mid Y=0) = \frac{P_{X,Y}(x,0)}{P_Y(0)}$$

$$P_X(x \mid Y=2) = \frac{P_{X,Y}(x,2)}{P_Y(2)}$$

	x = 0	x = 1	x = 2	x = 3
$P_x(x \mid Y = 0)$	$\frac{1}{19}$	0	$\frac{18}{19}$	0
$P_x(x \mid Y = 2)$	0	0	1	0







d) Determine a covariância entre X e Y.

Cov[X,Y] = E[(X - µx)(Y - µy)]
Cov[X,Y] = E[X,Y] - E[X]E[Y]

$$E[X] = 0.\frac{1}{64} + 1.\frac{9}{64} + 2.\frac{27}{64} + 3.\frac{27}{64} = 0 + \frac{9}{64} + \frac{54}{64} + \frac{81}{64} = \frac{144}{64} = \frac{9}{4}$$

$$E[X] = \frac{9}{4}$$

$$E[X] = (-1).\frac{3}{64} + 0.\frac{19}{64} + 1.\frac{33}{64} + 2.\frac{9}{64} = -\frac{3}{64} + 0 + \frac{33}{64} + \frac{18}{64} = \frac{48}{64} = \frac{3}{4}$$

$$E[X] = \frac{3}{4}$$

$$
\begin{aligned}
\mathrm{E[XY]} = {} & (0.(-1)).0 + (0.0).\frac{1}{64} + (0.1).0 + (0.2).0 + \\
& (1.(-1)).\frac{3}{64} + (1.0).0 + (1.1).\frac{3}{64} + (1.2).0 + \\
& (2.(-1)).0 + (2.0).\frac{9}{32} + (2.1).0 + (2.2).\frac{9}{64} + \\
& (3.(-1)).0 + (3.0).0 + (3.1).\frac{27}{64} + (3.2).0 =
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{E[XY]} = {} & 0 + 0 + 0 + 0 + \\
& -\frac{3}{64} + 0 + \frac{3}{64} + 0 + \\
& 0 + 0 + 0 + \frac{36}{64} + \\
& 0 + 0 + \frac{81}{64} + 0 =
\end{aligned}
$$

$$\mathrm{E[XY]} = \frac{36}{64} + \frac{81}{64} = \frac{117}{64}$$

Tendo os fatores acima calculados, basta achar a covariância:

$$
\begin{aligned}
\mathrm{Cov[X,Y]} &= \mathrm{E[XY]} - \mathrm{E[X]E[Y]} \\
&= \frac{117}{64} - \left(\frac{9}{4}.\frac{3}{4}\right)
\end{aligned}
$$

$$\mathrm{Cov[X,Y]} = \frac{117}{64} - \frac{27}{16}$$

$$\mathrm{Cov[X,Y]} = \frac{9}{64}$$