INSTITUTO FEDERAL DE EDUCAÇÃO, CIÊNCIA E TECNOLOGIA DE SANTA CATARINA
CAMPUS SÃO JOSÉ
Componente Curricular: Processos Estocásticos
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Data: 02/07/2021

Avaliação 1

8. Considere uma variável aleatória X definida através do seguinte experimento probabilístico. Um dado honesto é lançado.

- Se o resultado for 1 ou 4, então $X \sim \text{Unif}([0,2])$;
- Se o resultado for 2 ou 5, então $X \sim \text{Bern}\left(\frac{2}{3}\right)$;
- Se o resultado for 3 ou 6, então $X \sim \text{Unif}([-2,2])$;

(a) Determine e esboce a PDF de X.
(b) Determine e esboce a CDF de X.
(c) Determine a média de X.
(d) Determine $\Pr[X \leq 0]$.

Resolução:

Sendo um dado honesto lançado uma única vez, há $\frac{1}{6}$ de probabilidade de sortear um dos números de 1 à 6:

S = {1,2,3,4,5,6}

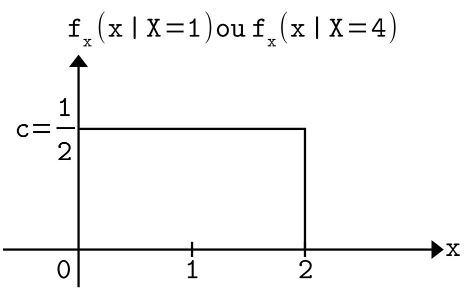
$$\Pr[1 \text{ ou } 4] = \Pr[1] + \Pr[4] = \frac{1}{6} + \frac{1}{6} = \frac{2}{6}$$

$$\Pr[2 \text{ ou } 5] = \Pr[2] + \Pr[5] = \frac{1}{6} + \frac{1}{6} = \frac{2}{6}$$

$$\Pr[3 \text{ ou } 6] = \Pr[3] + \Pr[6] = \frac{1}{6} + \frac{1}{6} = \frac{2}{6}$$

Temos que a probabilidade de termos uma das 3 situações acima é de $\frac{2}{6}$. Tendo que:

$X \sim \text{Unif}([\overset{a}{0},\overset{b}{2}])$ caso o resultado seja 1 ou 4.



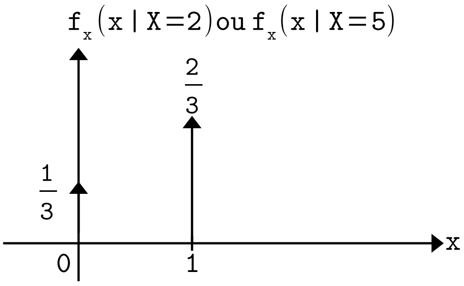
$$c = \frac{1}{b-a}$$

$$c = \frac{1}{2-0} = \frac{1}{2} \qquad \frac{1}{2}[0 \leq x \leq 2]$$

Média:

$$E[x] = \int_{-\infty}^{\infty} x f_x(x)dx = \int_0^2 x\frac{1}{2-0}dx = \frac{1}{2-0}\int_0^2 x\,dx = \frac{1}{2-0}\,.\,\frac{x^2}{2}\Bigg|_{x=0}^{x=2} = \frac{2^2-0^2}{2(2-0)} = \frac{4}{4} = 1$$

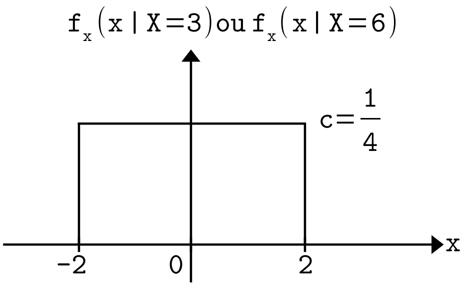
$X \sim \text{Bern}\left(\frac{2}{3}\right)$ caso o resultado seja 2 ou 5.



$$\frac{1}{3}\delta(x) + \frac{2}{3}\delta(x-1)$$

$$E[X] = \frac{2}{3}.1 + \frac{1}{3}.0 = \frac{2}{3}$$

$X \sim \text{Unif}([\overset{a}{-2},\overset{b}{2}])$ caso o resultado seja 3 ou 6.



$$c = \frac{1}{b-a} = \frac{1}{2-(-2)} = \frac{1}{2+2} = \frac{1}{4}$$

$$\frac{1}{4}[-2 \le x \le 2]$$

Média:

$$E[x] = \int_{-\infty}^{\infty} x f_x(x)dx = \int_{-2}^{2} x\frac{1}{2-(-2)}dx = \frac{1}{4}\int_{-2}^{2} x\,dx = \frac{1}{4}\,.\,\frac{x^2}{2}\Bigg|_{x=-2}^{x=2} = \frac{1}{4}\left[\frac{2^2}{2}-\frac{(-2)^2}{2}\right] = \frac{1}{4}\left[\frac{4}{2}-\frac{4}{2}\right] = 0$$

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a) Determine e esboce a PDF de X.

PDF pelo teorema da probabilidade total:

$$f_x(x) = \underbrace{f_x(x\,|\,U=1)}_{\sim \text{Unif}(0,2)}\underbrace{\Pr[U=1]}_{\frac{1}{6}} + \underbrace{f_x(x\,|\,U=4)}_{\sim \text{Unif}(0,2)}\underbrace{\Pr[U=4]}_{\frac{1}{6}} + \underbrace{f_x(x\,|\,U=2)}_{\sim \text{Bern}\left(\frac{2}{3}\right)}\underbrace{\Pr[U=2]}_{\frac{1}{6}} + \underbrace{f_x(x\,|\,U=5)}_{\sim \text{Bern}\left(\frac{2}{3}\right)}\underbrace{\Pr[U=5]}_{\frac{1}{6}} +$$

$$+ \underbrace{f_x(x\,|\,U=3)}_{\sim \text{Unif}(-2,2)}\underbrace{\Pr[U=3]}_{\frac{1}{6}} + \underbrace{f_x(x\,|\,U=6)}_{\sim \text{Unif}(-2,2)}\underbrace{\Pr[U=6]}_{\frac{1}{6}}$$

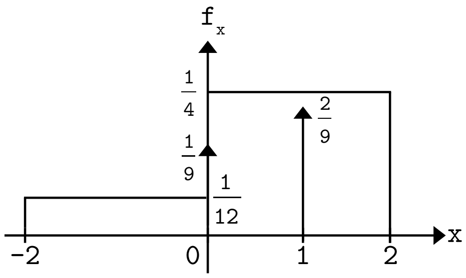
$$f_x(x) = \frac{1}{2}[0 \le x \le 2]\,.\,\frac{1}{6} + \frac{1}{2}[0 \le x \le 2]\,.\,\frac{1}{6} + \left(\frac{1}{3}\delta(X)+\frac{2}{3}\delta(x-1)\right).\,\frac{1}{6} + \left(\frac{1}{3}\delta(X)+\frac{2}{3}\delta(x-1)\right).\,\frac{1}{6} +$$

$$+ f_x(x)=\frac{1}{4}[-2 \le x \le 2]\,.\,\frac{1}{6} + f_x(x)=\frac{1}{4}[-2 \le x \le 2]\,.\,\frac{1}{6}$$

$$f_x(x) = \frac{1}{12}[0 \le x \le 2] + \frac{1}{12}[0 \le x \le 2] + \frac{1}{18}\delta(x) + \frac{1}{9}\delta(x-1) + \frac{1}{18}\delta(x) + \frac{1}{9}\delta(x-1) + \frac{1}{24}[-2 \le x \le 2] +$$

$$+ \frac{1}{24}[-2 \le x \le 2]$$

$$f_x(x) = \frac{1}{6}[0 \le x \le 2] + \frac{1}{9}\delta(x) + \frac{2}{9}\delta(x-1) + \frac{1}{12}[-2 \le x \le 2]$$



b) Determine e esboce a CDF de X.

$$F_x(x) = \Pr[X \le x] = \int_{-\infty}^{x^+} f_x(u)\,du$$

$$F_x(x) = \underbrace{\int_{-\infty}^{-2} 0\,dx}_{0} + \underbrace{\int_{-2}^{0^-} \frac{1}{12}dx}_{\frac{1}{6}} + \underbrace{\int_{0^-}^{0^+} \frac{1}{9}\delta(x)dx}_{\frac{1}{9}} + \underbrace{\int_{0^+}^{1^-} \frac{1}{4}dx}_{\frac{1}{4}} + \underbrace{\int_{1^-}^{1^+} \frac{2}{9}\delta(x-1)dx}_{\frac{2}{9}} + \underbrace{\int_{1^+}^{2} \frac{1}{4}dx}_{\frac{1}{4}} + \underbrace{\int_{2}^{x} 0\,dx}_{0}$$

Caso x < -2:

$$F_x(x) = \underbrace{\int_{-\infty}^{-2} 0\,dx}_{0}$$

Caso -2 < x < 0:

$$F_x(x) = \underbrace{\int_{-\infty}^{-2} 0\,dx}_{0} + \underbrace{\int_{-2}^{x} \frac{1}{12}dx}_{\frac{x+2}{12}}$$

Caso x = 0:

$$F_x(x) = \underbrace{\int_{-\infty}^{-2} 0\,dx}_{0} + \underbrace{\int_{-2}^{0^-} \frac{1}{12}dx}_{\frac{1}{6}} + \underbrace{\int_{0^-}^{x} \frac{1}{9}\delta(x)dx}_{\frac{1}{9}}$$

Caso 0 < x < 1:

$$F_x(x) = \underbrace{\int_{-\infty}^{-2} 0\,dx}_{0} + \underbrace{\int_{-2}^{0^-} \frac{1}{12}dx}_{\frac{1}{6}} + \underbrace{\int_{0^-}^{0^+} \frac{1}{9}\delta(x)dx}_{\frac{1}{9}} + \underbrace{\int_{0^+}^{x} \frac{1}{4}dx}_{\frac{1}{4}x}$$

Caso x = 1:

$$F_x(x) = \underbrace{\int_{-\infty}^{-2} 0\,dx}_{0} + \underbrace{\int_{-2}^{0^-} \frac{1}{12}dx}_{\frac{1}{6}} + \underbrace{\int_{0^-}^{0^+} \frac{1}{9}\delta(x)dx}_{\frac{1}{9}} + \underbrace{\int_{0^+}^{1^-} \frac{1}{4}dx}_{\frac{1}{4}} + \underbrace{\int_{1^-}^{x} \frac{2}{9}\delta(x-1)dx}_{\frac{2}{9}}$$

Caso 1 < x < 2:

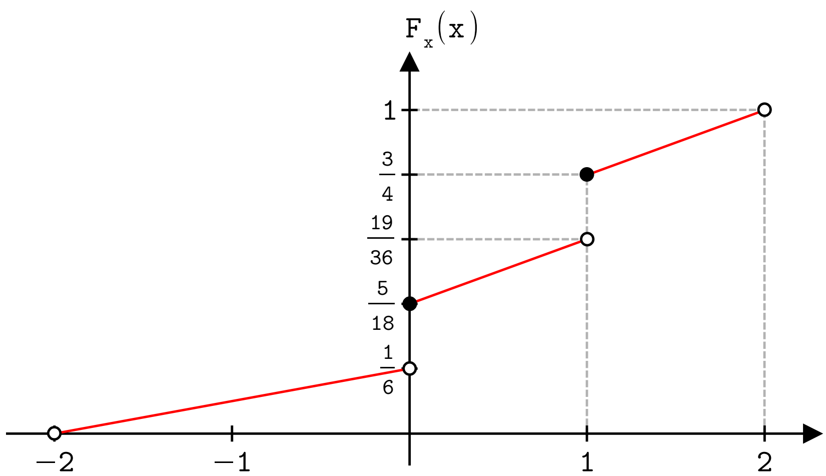
$$F_x(x) = \underbrace{\int_{-\infty}^{-2} 0\,dx}_{0} + \underbrace{\int_{-2}^{0^-} \frac{1}{12}dx}_{\frac{1}{6}} + \underbrace{\int_{0^-}^{0^+} \frac{1}{9}\delta(x)dx}_{\frac{1}{9}} + \underbrace{\int_{0^+}^{1^-} \frac{1}{4}dx}_{\frac{1}{4}} + \underbrace{\int_{1^-}^{x} \frac{2}{9}\delta(x-1)dx}_{\frac{2}{9}} + \underbrace{\int_{1^+}^{x} \frac{1}{4}dx}_{\frac{1}{4}(x-1)}$$

Caso x > 2:

$$F_x(x) = \underbrace{\int_{-\infty}^{-2} 0\,dx}_{0} + \underbrace{\int_{-2}^{0^-} \frac{1}{12}dx}_{\frac{1}{6}} + \underbrace{\int_{0^-}^{0^+} \frac{1}{9}\delta(x)dx}_{\frac{1}{9}} + \underbrace{\int_{0^+}^{1^-} \frac{1}{4}dx}_{\frac{1}{4}} + \underbrace{\int_{1^-}^{x} \frac{2}{9}\delta(x-1)dx}_{\frac{2}{9}} + \underbrace{\int_{1^+}^{2} \frac{1}{4}dx}_{\frac{1}{4}} + \underbrace{\int_{2}^{x} 0\,dx}_{0}$$

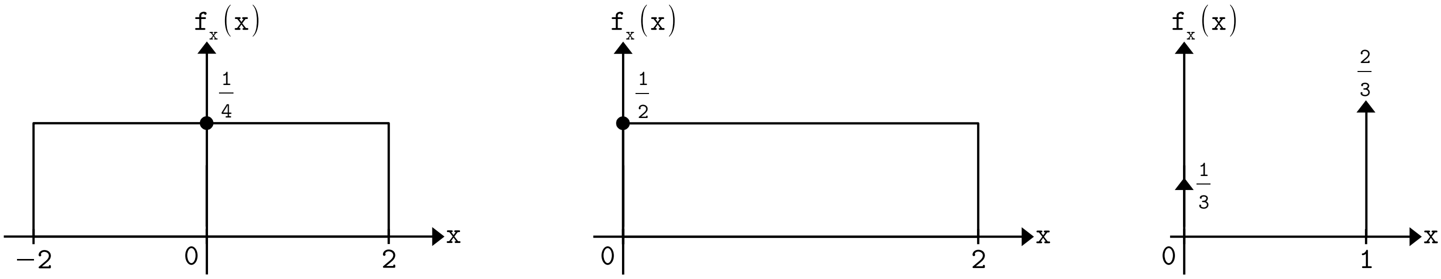
Sumário:

$$F_x(x) = \begin{cases} 0\,, & x < 0 \\ \frac{x+2}{12}\,, & -2 < x < 0 \\ \frac{5}{18}\,, & x = 0 \\ \frac{5}{18}+\frac{1}{4}x\,, & 0 < x < 1 \\ \frac{3}{4}\,, & x = 1 \\ \frac{3}{4}+\frac{1}{4}(x-1)\,, & 1 < x < 2 \\ 1\,, & x > 2 \end{cases} = \begin{cases} 0\,, & x < 0 \\ \frac{x+2}{12}\,, & -2 < x < 0 \\ \frac{5}{18}+\frac{1}{4}x\,, & 0 \le x \le 1 \\ \frac{3}{4}+\frac{1}{4}(x-1)\,, & 1 \le x \le 2 \\ 1\,, & x > 2 \end{cases}$$



c) Determine a média (valor esperado) de X.

Pelo teorema da probabilidade total:



$$E[x] = \underbrace{E[x \mid U=1 \vee 4]}_{1} \cdot \underbrace{\Pr[U=1 \vee 4]}_{\frac{2}{6}} + \underbrace{E[x \mid U=2 \vee 5]}_{\frac{2}{3}} \cdot \underbrace{\Pr[U=2 \vee 5]}_{\frac{2}{6}} + \underbrace{E[x \mid U=3 \vee 6]}_{0} \cdot \underbrace{\Pr[U=3 \vee 6]}_{\frac{2}{6}}$$

$$E[x] = +\frac{2}{6} + \frac{2}{9} = \frac{5}{9}$$

d) Determine Pr[X ≤ 0].

$$\Pr[X \leq 0] = \int_{-\infty}^{0^+} f_x(x)\,dx$$

$$\Pr[X \leq 0] = \int_{-\infty}^{-2} 0\,dx + \int_{-2}^{0^-} \frac{1}{12}\,dx + \int_{0^-}^{0^+} \frac{1}{9}\,\delta(x)\,dx$$

$$\Pr[X \leq 0] = 0 + \frac{1}{6} + \frac{1}{9}$$

$$\Pr[X \leq 0] = \frac{5}{18}$$